Chapter 6

L^p Spaces

6.1 Basic properties

The Lebesgue spaces, also known as the L^p spaces, constitute a rich source of examples and counter-examples in functional analysis. They also form an important class of function spaces when studying the applications of mathematical analysis. In this chapter, we will study the important properties of these spaces from the functional analytic point of view.

Let $(X, \mathcal{S}, \mu)$ be a measure space (cf. Section 1.3). Let $f : X \to \mathbb{R}$ be a real valued measurable function defined on X. Let $1 \leq p < \infty$. We define

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} \tag{6.1.1}$$

and we say that f is **p-integrable** (integrable, if $p = 1$ and square integrable, if $p = 2$) if $\|f\|_p < \infty$. Next, let $M > 0$. We set

$$\{|f| > M\} = \{x \in X \mid |f(x)| > M\}.$$

We now define

$$\|f\|_\infty = \inf\{M > 0 \mid \mu(\{|f| > M\}) = 0\} \tag{6.1.2}$$

and we say that f is *essentially bounded* if $\|f\|_\infty < \infty$.

Proposition 6.1.1 *(**Hölder's Inequality**) Let $1 \leq p < \infty$ and let p^* be the conjugate exponent. If f is p-integrable and g is p^*-integrable (essentially bounded, if $p = 1$), then*

$$\int_X |fg| \, d\mu \leq \|f\|_p \|g\|_{p^*}. \tag{6.1.3}$$

Proof: If $p = 1$, then $p^* = \infty$. Then

$$|f(x)g(x)| \;\leq\; |f(x)|.\|g\|_\infty$$

for almost every $x \in X$ and then (6.1.3) follows on integrating this inequality over X.

Let us now assume that $1 < p < \infty$ so that $1 < p^* < \infty$ as well. The relation (6.1.3) is trivially true if $\|f\|_p$ (respectively, $\|g\|_{p^*}$) equals zero, for then f (respectively, g) will be equal to zero almost everywhere. So we assume further that $\|f\|_p \neq 0$ and that $\|g\|_{p^*} \neq 0$. Then (cf. Lemma 2.2.1)

$$|f(x)g(x)| \;\leq\; \frac{1}{p}|f(x)|^p + \frac{1}{p^*}|g(x)|^{p^*}$$

for all $x \in X$. Assume now that $\|f\|_p = \|g\|_{p^*} = 1$. Then, integrating the above inequality over X, we get

$$\int_X |fg| \; d\mu \;\leq\; \frac{1}{p} + \frac{1}{p^*} \;=\; 1.$$

For the general case, apply the preceding result to the functions $f/\|f\|_p$ and $g/\|g\|_{p^*}$ to get (6.1.3). ■

Remark 6.1.1 When $p = p^* = 2$, once again (6.1.3) is known as the **Cauchy-Schwarz inequality.** ■

Proposition 6.1.2 (**Minkowski's Inequality**) *Let* $1 \leq p \leq \infty$. *Let f and g be p-integrable. Then $f + g$ is also p-integrable and*

$$\|f + g\|_p \;\leq\; \|f\|_p + \|g\|_p. \tag{6.1.4}$$

Proof: We assume that $\|f + g\|_p \neq 0$, since, otherwise, the result is trivially true. Since the function $t \mapsto |t|^p$ is convex for $1 \leq p < \infty$, we have that

$$|f(x) + g(x)|^p \;\leq\; 2^{p-1}(|f(x)|^p + |g(x)|^p)$$

from which it follows that $f + g$ is also p-integrable. Thus, if $1 < p < \infty$, we have

$$\int_X |f + g|^p \; d\mu \;\leq\; \int_X |f + g|^{p-1}|f| \; d\mu + \int_X |f + g|^{p-1}|g| \; d\mu.$$

We apply Hölder's inequality to each of the terms on the right-hand side. Notice that $|f(x) + g(x)|^{(p-1)p^*} = |f(x) + g(x)|^p$ by the definition of p^*. Thus $|f + g|^{p-1}$ is p^*-integrable and

$$\| \, |f + g|^{p-1}\|_{p^*} \;=\; \|f + g\|_p^{\frac{p}{p^*}}.$$

Thus,

$$\|f+g\|_p^p \le \|f+g\|_p^{\frac{p}{p^*}}(\|f\|_p + \|g\|_p).$$

Dividing both sides by $\|f+g\|_p^{\frac{p}{p^*}}$ and using, once again, the definition of p^*, we get (6.1.4). The cases where $p = 1$ and $p = \infty$ follow trivially from the inequality

$$|f(x) + g(x)| \le |f(x)| + |g(x)|.$$

This completes the proof. ∎

It is now easy to see that the space of all p-integrable functions ($1 \le p < \infty$) and that of all essentially bounded functions are vector spaces and that the map $f \mapsto \|f\|_p$ for $1 \le p \le \infty$ verifies all the properties of the norm, except that $\|f\|_p = 0$ *does not* imply that $f = 0$, but that $f = 0$ *almost everywhere* (a.e.; cf. Section 1.3).

Given two measurable functions f and g, we say that $f \sim g$ if $f = g$ almost everywhere, *i.e.* $f(x) = g(x)$ everywhere, except over a subset of measure zero. This defines an equivalence relation. If $f \sim g$, then for $1 \le p \le \infty$, we have that $\|f\|_p = \|g\|_p$. Further the set of all equivalence classes forms a vector space with respect to pointwise addition and scalar multiplication defined via arbitrary representatives of equivalence classes. In other words, if $f_1 \sim f_2$ and if $g_1 \sim g_2$, then $f_1 + g_1 \sim f_2 + g_2$ and, for any scalar α, we also have $\alpha f_1 \sim \alpha f_2$ and so on. Since $\|.\|_p$ is also constant on any equivalence class, we can define the 'norm' of an equivalence class via any representative function of that class. Further, if $\|f\|_p = 0$, then f will belong to the equivalence class of the function which is identically zero. Thus the set of all equivalence classes, with $\|\cdot\|_p$, becomes a normed linear space.

Definition 6.1.1 *Let (X, S, μ) be a measure space. Let $1 \le p < \infty$. The space of all equivalence classes, under the equivalence relation defined by equality of functions almost everywhere, of all p-integrable functions is a normed linear space with the norm of an equivalence class being the $\|.\|_p$-'norm' of any representative of that class. This space is denoted $L^p(\mu)$. The space of all equivalence classes of all essentially bounded functions with the norm of an equivalence class being defined as the $\|.\|_\infty$-'norm' of any representative of that class, is denoted $L^\infty(\mu)$.* ∎

While we may often talk of 'L^p-functions' we must keep in mind that we are really talking about equivalence classes of functions and that we carry out computations via representatives of those equivalence classes.

Notation We will denote elements of $L^p(\mu)$ by lower case Roman letters in sanserif font and a generic representative of the equivalence class it represents by the same lower case Roman letter in italicised font. Thus if we have $\mathsf{f} \in L^p(\mu)$, a generic representative will be f and so, for instance,

$$\|\mathsf{f}\|_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$.
Similarly, the equivalence class of a function f will be denoted by f.

Notation If $X = \Omega$, an open set of $\mathbb{R}^n$ provided with the Lebesgue measure, then the corresponding spaces $L^p(\mu)$ will be denoted $L^p(\Omega)$. In particular, if $\mathbb{R}$ is provided with the Lebesgue measure and if (a, b) is an interval, where $-\infty \leq a < b \leq +\infty$, then the L^p spaces on (a, b) will be denoted $L^p(a, b)$.

Example 6.1.1 Let $X = \{1, 2, \cdots, n\}$. Let $\mathcal{S}$ be the collection of all subsets of X and let μ be the counting measure (cf. Example 1.3.1). Then a measurable function can be identified with an n-tuple $(a_1, a_2, \cdots, a_n)$. In this case $L^p(\mu) = \ell_p^n$. Notice that in this example, equality almost everywhere is the same as equality everywhere and so every equivalence class is a singleton. ■

Example 6.1.2 Let $X = \mathbb{N}$, the set of all natural numbers and let $\mathcal{S}$ be the collection of all subsets of X. Let μ be the counting measure. In this case, functions are identified with real sequences and $L^p(\mu) = \ell_p$. Again, in this example, equivalence classes are just singletons. ■

Proposition 6.1.3 *Let $(X, \mathcal{S}, \mu)$ be a finite measure space, i.e. $\mu(X) < \infty$. Then*

$$L^p(\mu) \subset L^q(\mu)$$

with the inclusion being continuous, whenever $1 \leq q \leq p$.

Proof: The result is trivial if $p = \infty$. Let $1 \leq q < p < \infty$ and let

$\mathsf{f} \in L^p(\mu)$. Then, by Hölder's inequality, we have

$$\begin{aligned} \int |f|^q \, d\mu &\le \left(\int_X (|f|^q)^{\frac{p}{q}} \, d\mu\right)^{\frac{q}{p}} \left(\int_X d\mu\right)^{1-\frac{q}{p}} \\ &= \left(\int_X |f|^p \, d\mu\right)^{\frac{q}{p}} (\mu(X))^{1-\frac{q}{p}} \\ &= \|\mathsf{f}\|_p^q . (\mu(X))^{1-\frac{q}{p}} \end{aligned}$$

which yields

$$\|\mathsf{f}\|_q \le C \, \|\mathsf{f}\|_p$$

where

$$C = (\mu(X))^{\frac{1}{q}-\frac{1}{p}}.$$

This completes the proof. ■

Example 6.1.3 No such inclusions hold in infinite measure spaces. For instance, the sequence $(\frac{1}{n})$ belongs to ℓ_2 but not to ℓ_1. ■

Example 6.1.4 Nothing can be said about the reverse inclusions. For example, if $f(x) = 1/\sqrt{x}$, then $\mathsf{f} \in L^1(0,1)$ but $\mathsf{f} \notin L^2(0,1)$. However, we know that (cf. Exercise 2.25) $\ell_p \subset \ell_q$ for all $1 \le p < q \le \infty$. ■

Theorem 6.1.1 *Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $1 \le p \le \infty$. Then $L^p(\mu)$ is a Banach space.*

Proof: Case 1. Let $1 \le p < \infty$. Let $\{\mathsf{f}_n\}$ be a Cauchy sequence in $L^p(\mu)$. It is enough to show that there exists a convergent subsequence (why?). Choose a subsequence such that

$$\|f_{n_k} - f_{n_{k+1}}\|_p \le \frac{1}{2^k}.$$

Set

$$g_n(x) = \sum_{k=1}^{n} |f_{n_{k+1}}(x) - f_{n_k}(x)|$$

and

$$g(x) = \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|.$$

Then

$$\|g_n\|_p \le 1.$$

It follows that $g_n(x) \to g(x)$ and, by the monotone convergence theorem (cf. Theorem 1.3.1), we see that $\|g\|_p \leq 1$. In particular, $g(x) < \infty$ almost everywhere. Further, if $k \geq l \geq 2$, we have

$$\begin{aligned} |f_{n_k}(x) - f_{n_l}(x)| &\leq |f_{n_k}(x) - f_{n_{k-1}}(x)| + \cdots + |f_{n_{l+1}}(x) - f_{n_l}(x)| \\ &\leq g(x) - g_{l-1}(x). \end{aligned}$$

Thus, it follows that, for almost every $x \in X$, $\{f_{n_k}(x)\}$ is a Cauchy sequence and converges almost everywhere to a finite limit $f(x)$ and that, for such x,

$$|f(x) - f_{n_k}(x)| \leq g(x)$$

for $k \geq 2$. Set $f = 0$ elsewhere, which is a set of measure zero. It then follows that f is p-integrable. Further, $|f_{n_k}(x) - f(x)|^p \to 0$ almost everywhere and is bounded by $|g(x)|^p$ which is integrable. Hence, by the dominated convergence theorem (cf. Theorem 1.3.3), we deduce that $\|f_{n_k} - f\|_p \to 0$. Thus we have that

$$\mathsf{f}_{n_k} \to \mathsf{f}$$

in $L^p(\mu)$.

Case 2. $p = \infty$. Let $\{\mathsf{f}_n\}$ be Cauchy in $L^\infty(\mu)$. Then, for each k, there exists a positive integer N_k such that

$$\|\mathsf{f}_m - \mathsf{f}_n\|_\infty < \frac{1}{k}$$

for all $m, n \geq N_k$. Thus, there exists a set E_k of measure zero, such that

$$|f_m(x) - f_n(x)| \leq \frac{1}{k}$$

for all $m, n \geq N_k$ and for all $x \in X \backslash E_k$. Setting $E = \cup_{k=1}^\infty E_k$, we see that E is of measure zero and for all $x \in X \backslash E$, the sequence $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$. Thus, for all such x, $f_n(x) \to f(x)$. Passing to the limit as $m \to \infty$, we see that, for all $x \in X \backslash E$, and for all $n \geq N_k$,

$$|f(x) - f_n(x)| \leq \frac{1}{k}.$$

Hence, it follows that f is essentially bounded and that $\mathsf{f}_n \to \mathsf{f}$ in $L^\infty(\mu)$.

This completes the proof. ■

Corollary 6.1.1 *Let $(X, \mathcal{S}, \mu)$ be a measure space and let $\mathsf{f}_n \to \mathsf{f}$ in $L^p(\mu)$ for some $1 \le p \le \infty$. Then, there exists a subsequence f_{n_k} such that*
(i) $f_{n_k}(x) \to f(x)$ *almost everywhere.*
(ii) $|f_{n_k}(x)| \le h(x)$ *almost everywhere for some* $\mathsf{h} \in L^p(\mu)$.

Proof: The result is obvious in the case $p = \infty$. Let $1 \le p < \infty$. Then, as in the case of the preceding theorem, we have a subsequence $\{\mathsf{f}_{n_k}\}$ which converges to a function $\widetilde{\mathsf{f}}$ in $L^p(\mu)$and also such that $f_{n_k}(x) \to \widetilde{f}(x)$ almost everywhere. It is then clear that $\widetilde{\mathsf{f}} = \mathsf{f}$, *i.e.* $\widetilde{f} = f$ almost everywhere and this proves (i). To see (ii), take $h = \widetilde{f} + g$, where g is as in the proof of the preceding theorem. ■

6.2 Duals of L^p Spaces

In Chapter 3, we identified the dual of the space ℓ_p with ℓ_{p^*} where $1 \le p < \infty$ and p^* is the conjugate exponent of p. Similar results are true in more general L^p spaces.

Proposition 6.2.1 (**Clarkson's Inequality**) *Let $(X, \mathcal{S}, \mu)$ be a measure space and let $2 \le p < \infty$. Then if* f *and* $\mathsf{g} \in L^p(\mu)$,

$$\left\| \frac{1}{2}(\mathsf{f}+\mathsf{g}) \right\|_p^p + \left\| \frac{1}{2}(\mathsf{f}-\mathsf{g}) \right\|_p^p \le \frac{1}{2}\left(\|\mathsf{f}\|_p^p + \|\mathsf{g}\|_p^p \right). \tag{6.2.1}$$

Proof: Consider the function

$$\varphi(x) = (x^2+1)^{\frac{p}{2}} - x^p - 1$$

for $x \ge 0$. Then it is simple to check that $\varphi(0) = 0$ and that $\varphi'(x) > 0$ for $x > 0$ when $p \ge 2$. Thus, it follows that for all $x \ge 0$,

$$(x^2+1)^{\frac{p}{2}} \ge x^p + 1,$$

when $p \ge 2$. Hence, if α and β are positive real numbers, we have

$$(\alpha^2+\beta^2)^{\frac{p}{2}} \ge \alpha^p + \beta^p.$$

Combining this with the fact that the function $t \mapsto t^{\frac{p}{2}}$ is convex on the set $\{t \in \mathbb{R} \mid t \ge 0\}$, we get, for any $x \in X$ and for any f and $\mathsf{g} \in L^p(\mu)$,

$$\begin{aligned} \left|\frac{f(x)+g(x)}{2}\right|^p + \left|\frac{f(x)-g(x)}{2}\right|^p &\le \left(\left|\frac{f(x)+g(x)}{2}\right|^2 + \left|\frac{f(x)-g(x)}{2}\right|^2 \right)^{\frac{p}{2}} \\ &= \left(\frac{|f(x)|^2+|g(x)|^2}{2} \right)^{\frac{p}{2}} \\ &\le \tfrac{1}{2}(|f(x)|^p + |g(x)|^p) \end{aligned}$$

which yields (6.2.1) on integration over X. ■

Corollary 6.2.1 *Let $(X, \mathcal{S}, \mu)$ be a measure space. Then, the spaces $L^p(\mu)$ are reflexive when $2 \leq p < \infty$.*

Proof: Arguing as in Example 5.5.2, it is easy to see that (6.2.1) implies that $L^p(\mu)$ is uniformly convex when $2 \leq p < \infty$. The reflexivity now follows from Theorem 5.5.1. ■

Theorem 6.2.1 *(**Riesz Representation Theorem**) Let $(X, \mathcal{S}, \mu)$ be a measure space and let $1 < p < \infty$. Let p^* be the conjugate exponent. Then the dual of $L^p(\mu)$ is isometrically isomorphic to $L^{p^*}(\mu)$. In particular, the spaces $L^p(\mu)$ are reflexive for all $1 < p < \infty$.*

Proof: Step 1. Let $\mathbf{g} \in L^{p^*}(\mu)$. Define $T_{\mathbf{g}} : L^p(\mu) \to \mathbb{R}$ by

$$T_{\mathbf{g}}(\mathsf{f}) = \int_X fg \, d\mu$$

for $\mathsf{f} \in L^p(\mu)$. Clearly, $T_{\mathbf{g}}$ is a linear functional, and, by Hölder's inequality, it is continuous as well. In fact, we have

$$\|T_{\mathbf{g}}\| \leq \|\mathbf{g}\|_{p^*}.$$

Now, consider the function

$$f(x) = \begin{cases} |g(x)|^{p^*-2} g(x), & \text{if } g(x) \neq 0 \\ 0, & \text{if } g(x) = 0. \end{cases}$$

Then $|f|^p = |g|^{(p^*-1)p} = |g|^{p^*}$ so that f is p-integrable. Also

$$T_{\mathbf{g}}(f) = \int_X |g|^{p^*} \, d\mu$$

from which we deduce that

$$\|T_{\mathbf{g}}\| = \|\mathbf{g}\|_{p^*}.$$

Thus, the map $\mathbf{g} \mapsto T_{\mathbf{g}}$ is an isometry from $L^{p^*}(\mu)$ into $L^p(\mu)^*$. Hence its image is closed. It is enough now to show that the image is dense.

Step 2. We now show that $L^p(\mu)$ is reflexive for all $1 < p < \infty$. This has already been proved for $2 \leq p < \infty$. Thus $L^p(\mu)^*$ is also reflexive for such p and so is every closed subspace of this dual space. Thus, by the preceding step, $L^{p^*}(\mu)$ which is isometrically isomorphic to a closed subspace of the dual of $L^p(\mu)$ is also reflexive for $2 \leq p < \infty$. But then $1 < p^* \leq 2$. This proves that $L^p(\mu)$ is also reflexive when $1 < p \leq 2$. This establishes the claim.

Step 3. We are now in a position to show that the isometry $\mathbf{g} \mapsto T_{\mathbf{g}}$ from $L^{p^*}(\mu)$ to the dual space $L^p(\mu)^*$ is onto. As already observed, the image is a closed subspace and we now show that it is dense. Indeed, let $\varphi \in L^p(\mu)^{**}$ vanish on the image. We need to show that φ is the zero functional. Since all the $L^p(\mu)$ are reflexive, this means that there exists $\mathsf{f} \in L^p(\mu)$ such that, for all $\mathbf{g} \in L^{p^*}(\mu)$, we have

$$\int_X fg \, d\mu = 0.$$

Once again, choosing $g = |f|^{p-2}f$ (and equal to zero where f vanishes) we deduce that $\mathsf{f} = 0$. This completes the proof. ■

Remark 6.2.1 We have seen earlier that $\ell_1^* = \ell_\infty$. In the same way, it is true that for σ-finite measure spaces, we have $L^1(\mu)^* = L^\infty(\mu)$. However, the proof of this result relies on very measure theoretic arguments and we shall omit it. Nevertheless, in the next section, we will prove it for a very important class of L^1 spaces. ■

6.3 The Spaces $L^p(\Omega)$

In this section, we will study the properties of a very important class of L^p spaces defined on open sets in the Euclidean spaces $\mathbb{R}^N$.

Let $\Omega \subset \mathbb{R}^N$ be an open set. Consider the Lebesgue measure on this set. Then, as mentioned in Section 6.1, we will denote the corresponding L^p spaces by $L^p(\Omega)$.

In the sequel, if we say that a certain function space is contained in (respectively, is dense in), $L^p(\Omega)$, we will understand that we are talking about the set of all equivalence classes of functions in that space being contained in (respectively, being dense in) $L^p(\Omega)$.

Proposition 6.3.1 *Let S be the set of all simple functions which vanish outside a set of finite measure. Then S is dense in $L^p(\Omega)$ for $1 \leq p < \infty$.*

Proof: Let $\varphi \in S$. Since φ vanishes outside a set of finite measure, it is automatically p-integrable for $1 \leq p < \infty$. Let $f \geq 0$ be a p-integrable function. Then, there exists a sequence $\{\varphi_n\}$ of non-negative simple functions which increase to f (cf. Proposition 1.3.2). Since f is p-integrable, so is φ_n and so φ_n will also vanish outside a set of finite measure. Further

$$|\varphi_n(x) - f(x)|^p \leq 2^p |f(x)|^p$$

for $x \in \Omega$ and, since $|f|^p$ is integrable, it follows from the dominated convergence theorem that

$$\int_\Omega |\varphi_n - f|^p \, dx \ \rightarrow \ 0$$

as $n \rightarrow \infty$. If f is an arbitrary p-integrable function, then we have sequences $\{\varphi_n\}$ and $\{\psi_n\}$ of simple functions vanishing outside sets of finite measure and such that

$$\int_\Omega |\varphi_n - f^+|^p \, dx \ \rightarrow \ 0 \text{ and } \int_\Omega |\psi_n - f^-|^p \, dx \ \rightarrow \ 0.$$

Thus $\chi_n = \varphi_n - \psi_n$ is a simple function which vanishes outside a set of finite measure and

$$\int_\Omega |\chi_n - f|^p \, dx \ \rightarrow \ 0$$

as $n \rightarrow \infty$. This proves the result. ■

Theorem 6.3.1 *Let $1 \leq p < \infty$. Let $\Omega \subset \mathbb{R}^N$ be open. Then, $\mathcal{C}_c(\Omega)$, the space of all continuous functions with compact support contained in Ω, is dense in $L^p(\Omega)$.*

Proof: By the preceding proposition, we know that S is dense in $L^p(\Omega)$. Thus, given $\varphi \in S$, it is enough to show that it can be approximated (in the L^p-norm) as closely as we wish by a continuous function with compact support. Indeed, let $\varepsilon > 0$. By Lusin's theorem (cf. Royden [5]), we can find a continuous function g, with compact support, such that $g = \varphi$ except possibly on a set whose measure is less than ε and also such that $|g(x)| \leq \|\varphi\|_\infty$. Then

$$\int_\Omega |g - \varphi|^p \, dx \ \leq \ 2^p \|\varphi\|_\infty^p \varepsilon.$$

This shows that $\mathcal{C}_c(\Omega)$ is dense (with respect to the norm $\|.\|_p$) in S which, in turn, is dense in $L^p(\Omega)$. This proves the result. ■

Remark 6.3.1 In fact it can be shown that the space of infinitely differentiable functions with compact support contained in Ω is dense in $L^p(\Omega)$ for $1 \leq p < \infty$. For this we need to develop the theory of convolution of functions (cf. Theorem 6.3.3 below). For details, see Kesavan [3]. ■

Corollary 6.3.1 *Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $1 \leq p < \infty$. Then, $L^p(\Omega)$ is separable.*

Proof: Recall that, by the Weierstrass approximation theorem, a continuous functiuon on a compact set can be uniformly approximated by means of a polynomial and hence, by a polynomial with *rational* coefficients and such polynomials form a countable set.

We can write

$$\Omega \ = \ \cup_{n=1}^{\infty} \Omega_n$$

where $\Omega_n = \Omega \cap B(\mathbf{0}; n)$; here $B(\mathbf{0}; n)$ is the ball centred at the origin and with radius n in $\mathbb{R}^N$. Notice that Ω_n is bounded and is hence relatively compact, *i.e.* $\overline{\Omega_n}$ is compact.

Let $\varepsilon > 0$ and let f be p-integrable over Ω. Then, by the preceding theorem, we can find a continuous function g, with compact support such that $\|f - g\|_p < \varepsilon$. Since the support of g is compact, its support will lie in some Ω_n. Thus, we can find a polynomial p with rational coefficients such that, for all $x \in \Omega_n$,

$$|g(x) - \mathsf{p}(x)| \ < \ \frac{\varepsilon}{|\Omega_n|^{\frac{1}{p}}}$$

where $|\Omega_n|$ denotes the (Lebesgue) measure of Ω_n. Setting $\mathsf{p} = 0$ outside Ω_n, we then see that $\|g - \mathsf{p}\|_p < \varepsilon$ so that $\|f - \mathsf{p}\|_p < 2\varepsilon$. Thus any p-integrable function can be approximated in the norm $\|.\|_p$ by means of a function which vanishes outside some Ω_n and is equal to a polynomial with rational coefficients inside Ω_n. The collection of all such functions being countable, we deduce that $L^p(\Omega)$ is separable for $1 \leq p < \infty$. ■

Proposition 6.3.2 *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then, $L^\infty(\Omega)$ is not separable.*

Proof: Let $x \in \Omega$. Let $r = r(x) > 0$ be chosen such that the ball $B(x; r) \subset \Omega$. Define

$$\chi_x(y) \ = \ \begin{cases} 1, & \text{if } y \in B(x; r) \\ 0, & \text{otherwise.} \end{cases}$$

Set

$$U_x = \{\mathsf{f} \in L^\infty(\Omega) \mid \|\mathsf{f} - \chi_x\|_\infty < 1/2\}.$$

Then, for each $x \in \Omega$, U_x is a non-empty open subset of $L^\infty(\Omega)$. Further, if $x \neq y$, then $U_x \cap U_y = \emptyset$. For, if f belonged to their intersection, then on one hand, since $\mathsf{f} \in U_y$, we have that $|f(z)| < 1/2$ in a small neighbourhood of x. On the other hand, since f belongs to U_x, it follows that $|f(z)| > 1/2$ in a small neighbourhood of x, which is a contradiction.

Now let $E = \{\mathsf{f}_n\}$ be any countable set in $L^\infty(\Omega)$. If such a set were dense, then $E \cap U_x \neq \emptyset$ for each $x \in \Omega$. However, any f_n can belong to at most one such open set U_x since the sets U_x are pairwise disjoint. This is a contradiction since the number of open sets U_x is uncountable. Thus, no countable set in $L^\infty(\Omega)$ can be dense. ■

Definition 6.3.1 *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $f : \Omega \to \mathbb{R}$ be a measurable function. We say that f is* **locally integrable** *if $\int_K |f|\, dx < \infty$ for every compact set $K \subset \Omega$.* ■

We denote the set of all locally integrable functions on Ω by $L^1_{\text{loc}}(\Omega)$.

Proposition 6.3.3 *Let $f \in L^1_{\text{loc}}(\Omega)$ be such that*

$$\int_\Omega fg\, dx = 0$$

for all $g \in \mathcal{C}_c(\Omega)$. Then $f = 0$ almost everywhere in Ω.

Proof: Step 1. We first assume that f is integrable on Ω and that $|\Omega|$, the measure of Ω, is finite. Let $\varepsilon > 0$. Then, there exists a continuous function f_1, with compact support, such that $\|f - f_1\|_1 < \varepsilon$ (cf. Theorem 6.3.1). Thus, if $g \in \mathcal{C}_c(\Omega)$, we have

$$\left|\int_\Omega f_1 g\, dx\right| = \left|\int_\Omega (f_1 - f) g\, dx\right| \leq \varepsilon \|g\|_\infty. \tag{6.3.1}$$

Let

$$\begin{aligned} K_1 &= \{x \in \Omega \mid f_1(x) \geq \varepsilon\} \\ K_2 &= \{x \in \Omega \mid f_1(x) \leq -\varepsilon\}. \end{aligned}$$

Then, K_1 and K_2 are disjoint and compact sets (since f_1 is a continuous function with compact support) and by Urysohn's lemma, we can construct a continuous function h, also with compact support such that

$h \equiv 1$ on K_1 and $h \equiv -1$ on K_2. Further, we can also have $|h(x)| \leq 1$ for all $x \in \Omega$. Set $K = K_1 \cup K_2$. Then

$$\int_\Omega f_1 h \, dx = \int_{\Omega \backslash K} f_1 h \, dx + \int_K f_1 h \, dx,$$

whence, in view of (6.3.1), we have

$$\int_K |f_1| \, dx = \int_K f_1 h \, dx \leq \varepsilon + \int_{\Omega \backslash K} |f_1 h| \, dx \leq \varepsilon + \int_{\Omega \backslash K} |f_1| \, dx.$$

Since $|f_1(x)| \leq \varepsilon$ on $\Omega \backslash K$, we deduce that

$$\begin{aligned} \textstyle\int_\Omega |f_1| \, dx &= \textstyle\int_K |f_1| \, dx + \int_{\Omega \backslash K} |f_1| \, dx \\ &\leq \varepsilon + 2 \textstyle\int_{\Omega \backslash K} |f_1| \, dx \\ &\leq \varepsilon + 2\varepsilon |\Omega|. \end{aligned}$$

Thus,

$$\|f\|_1 \leq \|f - f_1\|_1 + \|f_1\|_1 \leq 2\varepsilon + 2\varepsilon|\Omega|.$$

Since ε is arbitrary, it follows that $f(x) = 0$ almost everywhere in Ω.

Step 2. In the general case, we again write $\Omega = \cup_{n=1}^\infty \Omega_n$ where $\Omega_n = \Omega \cap B(\mathbf{0}; n)$. Then applying the result of Step 1 to the restriction of f to Ω_n, denoted $f|_{\Omega_n}$, we get that $f|_{\Omega_n} = 0$ almost everywhere in Ω_n from which it immediately follows that $f = 0$ almost everywhere in Ω. ■

Let us now turn our attention to the space $L^1(\Omega)$.

Theorem 6.3.2 (**Riesz Representation Theorem**) *Let $\Omega \subset \mathbb{R}^N$ be an open set. The dual of the space $L^1(\Omega)$ is isometrically isomorphic to $L^\infty(\Omega)$.*

Proof: Step 1. There exists w $\in L^2(\Omega)$ such that $w(x) \geq \varepsilon_K > 0$ for all $x \in K$ for every compact subset K of Ω. Indeed, define $w(x) = \alpha_n > 0$ on the set

$$E_n = \{x \in \Omega \mid n \leq |x| < n+1\},$$

where $|x|$ denotes the Euclidean norm of the vector $x \in \mathbb{R}^N$. Now choose the constants α_n such that

$$\sum_{n=0}^\infty \alpha_n^2 |E_n| < \infty,$$

where $|E_n|$ denotes the (Lebesgue) measure of the set E_n. Then w has the required properties.

Step 2. Let $\varphi \in L^1(\Omega)^*$. Consider the mapping $\mathsf{f} \mapsto \varphi(\mathsf{wf})$ from $L^2(\Omega)$ into $\mathbb{R}$. Clearly, this defines a linear functional which, by Hölder's inequality, is also continuous. Thus, by the Riesz representation theorem (cf. Theorem 6.2.1) applied to the case $p = 2$, there exists $\mathsf{v} \in L^2(\Omega)$ such that

$$\varphi(\mathsf{wf}) = \int_\Omega fv\, dx$$

for all $\mathsf{f} \in L^2(\Omega)$. Thus, we have

$$\left|\int_\Omega fv\, dx\right| \leq \|\varphi\|.\|wf\|_1. \tag{6.3.2}$$

Step 3. Set $u(x) = v(x)/w(x)$ for $x \in \Omega$. Since w never vanishes, this is well defined and u is measurable. We claim that $\mathsf{u} \in L^\infty(\Omega)$ and that $\|\mathsf{u}\|_\infty \leq \|\varphi\|$. To see this it is sufficient to show that, for any constant $C > \|\varphi\|$, we have that the set

$$A = \{x \in \Omega \mid |u(x)| > C\}$$

is of measure zero.

Assume the contrary for some such $C > \|\varphi\|$.Then, there exists a subset B of A of finite and positive measure. Consider the function

$$f(x) = \begin{cases} +1, & \text{if } x \in B \text{ and } u(x) > 0, \\ -1, & \text{if } x \in B \text{ and } u(x) < 0, \\ 0, & \text{if} x \in \Omega \backslash B. \end{cases}$$

Clearly, f is square integrable (since the measure of B is finite) and we can use it in (6.3.2). We then get

$$\int_B |u|w\, dx \leq \|\varphi\| \int_B w\, dx$$

and, using the definition of A, which contains B, we get

$$C\int_B w\, dx \leq \|\varphi\| \int_B w\, dx$$

which is a contradiction to the choice of C, since $\int_B w\, dx > 0$.

Step 4. Thus we now have $\mathsf{u} \in L^\infty(\Omega)$ with $\|\mathsf{u}\|_\infty \leq \|\varphi\|$ such that, for all $\mathsf{f} \in L^2(\Omega)$,

$$\varphi(\mathsf{wf}) = \int_\Omega fuw \, dx.$$

Let $g \in \mathcal{C}_c(\Omega)$. Then, by choice of w, $f = g/w$ is square integrable and so, we get, for all g continuous with compact support in Ω,

$$\varphi(\mathbf{g}) = \int_\Omega ug \, dx. \tag{6.3.3}$$

Since $\mathcal{C}_c(\Omega)$ is dense in $L^1(\Omega)$, the above relation also holds for all $\mathbf{g} \in L^1(\Omega)$. Further, it follows that

$$|\varphi(\mathbf{g})| \leq \|\mathbf{g}\|_1 \|\mathsf{u}\|_\infty$$

for all $\mathbf{g} \in L^1(\Omega)$ from which we deduce that $\|\varphi\| \leq \|\mathsf{u}\|_\infty$.

Step 5. Thus, for every $\varphi \in L^1(\Omega)^*$, we have $\mathsf{u} \in L^\infty(\Omega)$ such that $\|\varphi\| = \|\mathsf{u}\|_\infty$ and such that (6.3.3) holds for all $\mathbf{g} \in L^1(\Omega)$. Such a u is unique as well. Indeed if we have two essentially bounded functions u_1 and u_2 such that

$$\int_\Omega g(u_1 - u_2) \, dx = 0$$

for all $\mathbf{g} \in L^1(\Omega)$, then it is in particular true for all $g \in \mathcal{C}_c(\Omega)$ and, since essentially bounded functions are locally integrable, it follows that (cf. Proposition 6.3.3) $u_1 - u_2 = 0$ almost everywhere, *i.e.* $\mathsf{u}_1 = \mathsf{u}_2$ in $L^\infty(\Omega)$.

Step 6. If $\mathsf{u} \in L^\infty(\Omega)$, then if we define T_u as a linear functional on $L^1(\Omega)$ via the right-hand side of (6.3.3), then we have just seen that $\mathsf{u} \mapsto T_\mathsf{u}$ is surjective and that it is an isometry between $L^\infty(\Omega)$ and $L^1(\Omega)^*$. This completes the proof. ■

Proposition 6.3.4 *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then, $L^1(\Omega)$ is not reflexive.*

Proof: Without loss of generality, assume that Ω contains the origin. Let n be sufficiently large so that the ball centred at the origin and of radius $1/n$, denoted B_n, is contained in Ω. Let $\alpha_n = |B_n|^{-1}$, where, as usual, $|B_n|$ denotes the (Lebesgue) measure of B_n. Let $f_n(x) = \alpha_n$ for all $x \in B_n$ and let it vanish on $\Omega \backslash B_n$. Then $\mathsf{f}_n \in L^1(\Omega)$ and $\|\mathsf{f}_n\|_1 = 1$ for all n. If $L^1(\Omega)$ were reflexive, then the sequence $\{\mathsf{f}_n\}$ would contain a weakly convergent subsequence (cf. Theorem 5.4.2), say $\{\mathsf{f}_{n_k}\}$. Let f be its weak limit. Then, for every $\mathsf{h} \in L^\infty(\Omega)$ we must have

$$\int_\Omega f_{n_k} h \, dx \;\to\; \int_\Omega f h \, dx. \tag{6.3.4}$$

Now choose $h \in \mathcal{C}_c(\Omega \backslash \{\mathbf{0}\})$. Then, for sufficiently large k, we have that

$$\int_\Omega f_{n_k} h \, dx \;=\; 0$$

(since the two functions in the integrand will then have disjoint supports) and so, it follows that, for all such h, we have $\int_\Omega fh \, dx = 0$. By Proposition 6.3.3, it then follows that $f(x) = 0$ almost everywhere in $\Omega \backslash \{\mathbf{0}\}$ and so $f(x) = 0$ almost everywhere on Ω as well. On the other hand, if we choose $h(x) = 1$ for all $x \in \Omega$ in (6.3.4), we get $\int_\Omega f \, dx = 1$, which is a contradiction. Thus, $L^1(\Omega)$ is not reflexive. ■

Corollary 6.3.2 *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then, $L^\infty(\Omega)$ is not reflexive.*

Proof: Since $L^1(\Omega)^* \cong L^\infty(\Omega)$, the result follows immediately from the preceding proposition (cf. Corollary 5.3.3). ■

To sum up, we have that $L^p(\Omega)^* \cong L^{p^*}(\Omega)$ for $1 \le p < \infty$. The spaces $L^p(\Omega)$ are separable for $1 \le p < \infty$ and reflexive for $1 < p < \infty$. The space $L^\infty(\Omega)$ is neither separable nor reflexive.

We conclude by proving an important inequality.

Theorem 6.3.3 *(**Young's Inequality**) Let $1 < p < \infty$. Let $\mathsf{f} \in L^1(\mathbb{R}^N)$ and let $\mathsf{g} \in L^p(\mathbb{R}^N)$. Then the map*

$$x \;\mapsto\; \int_{\mathbb{R}^N} f(y) g(x-y) \, dy$$

is well defined almost everywhere in $\mathbb{R}^N$. The function thus defined is denoted $\mathsf{f} * \mathsf{g}$ and is called the convolution *of* f *and* g. *Further, $\mathsf{f} * \mathsf{g} \in L^p(\mathbb{R}^N)$ and we also have*

$$\|\mathsf{f} * \mathsf{g}\|_p \;\le\; \|\mathsf{f}\|_1 \|\mathsf{g}\|_p. \tag{6.3.5}$$

Proof: Let $\mathsf{h} \in L^{p^*}(\mathbb{R}^N)$, where p^* is the conjugate exponent of p. The function $(x, y) \mapsto f(y) g(x-y) h(x)$ is measurable on $\mathbb{R}^N \times \mathbb{R}^N$; consider the iterated integral

$$I \;=\; \int_{\mathbb{R}^N_x} \int_{\mathbb{R}^N_y} |f(y) g(x-y) h(x)| \, dy \, dx.$$

Since the Lebesgue measure is translation invariant, we get that

$$\begin{aligned} I &= \int_{\mathbb{R}_y^N} |f(y)| \left(\int_{\mathbb{R}_x^N} |g(x-y)h(x)| \, dx \right) dy \\ &\leq \|\mathsf{g}\|_p \|\mathsf{h}\|_{p^*} \int_{\mathbb{R}^N} |f(y)| \, dy \\ &= \|\mathsf{f}\|_1 \|\mathsf{g}\|_p \|\mathsf{h}\|_{p^*} < \infty. \end{aligned}$$

Thus by Fubini's theorem, the integral

$$\int_{\mathbb{R}^N} f(y)g(x-y)h(x) \, dy$$

exists for almost all $x \in \mathbb{R}^N$. Let us choose $\mathsf{h} \in L^{p^*}(\mathbb{R}^N)$ such that $h(x) \neq 0$ for all $x \in \mathbb{R}^N$. For example, we can choose $h(x) = \exp(-|x|^2)$. Thus, it follows that the integral

$$\int_{\mathbb{R}^N} f(y)g(x-y) \, dy$$

exists for almost all $x \in \mathbb{R}^N$ and so the convolution $\mathsf{f} * \mathsf{g}$ is well defined. Further, by the above computation it follows that the map

$$\mathsf{h} \mapsto \int_{\mathbb{R}^N} h(x)(f*g)(x) \, dx$$

is a continuous linear functional on $L^{p^*}(\mathbb{R}^N)$ whose norm is bounded by $\|\mathsf{f}\|_1 \|\mathsf{g}\|_p$. It follows from the Riesz representation theorem that $\mathsf{f} * \mathsf{g} \in L^p(\mathbb{R}^N)$ and that (6.3.5) holds. ■

Remark 6.3.2 By a simple change of variable it is easy to see that we can also write the convolution of f and g as

$$(f*g)(x) = \int_{\mathbb{R}^N} f(x-y)g(y) \, dy.$$

Remark 6.3.3 The result of Theorem 6.3.3 is valid for the case $p = 1$ as well. The proof of this fact is left as an exercise (cf. Exercise 6.16). ■

6.4 The Spaces $W^{1,p}(a,b)$

In this section, we will study a very special case of a class of function spaces called *Sobolev spaces.*

Throughout this section, we assume that (a,b) is a finite interval in $\mathbb{R}$ and that $1 \leq p < \infty$. We will denote by $\mathcal{D}(a,b)$ the space of infinitely differentiable functions with compact support contained in the interval (a,b). Recall that (cf. Remark 6.3.1) $\mathcal{D}(a,b)$ is dense in $L^p(a,b)$ for $1 \leq p < \infty$.

Lemma 6.4.1 *Let* $\mathsf{f} \in L^p(a,b)$. *Assume that there exists* $\mathsf{g} \in L^p(a,b)$ *such that, for all* $\varphi \in \mathcal{D}(a,b)$, *we have*

$$\int_a^b f\varphi' \, dx \;=\; -\int_a^b g\varphi \, dx. \tag{6.4.1}$$

Then such a g *is unique.*

Proof: If there were two functions g_1 and g_2 satisfying (6.4.1) for a given f, then

$$\int_a^b (g_1 - g_2)\varphi \, dx \;=\; 0$$

for all $\varphi \in \mathcal{D}(a,b)$. Since $g_1 - g_2$ is locally integrable, it now follows that $g_1(x) = g_2(x)$ almost everywhere (cf. Proposition 6.3.3). ■

Definition 6.4.1 *Let* $(a,b) \subset \mathbb{R}$ *be a finite interval and let* $1 \leq p < \infty$. *The Sobolev space* $W^{1,p}(a,b)$ *is given by*

$$W^{1,p}(a,b) \;=\; \{\mathsf{f} \in L^p(a,b) \mid \textit{there exists } \mathsf{g} \in L^p(a,b) \textit{ satisfying (6.4.1)}\}.$$

Further, we define

$$\|\mathsf{f}\|_{1,p} \;=\; (\|\mathsf{f}\|_p^p + \|\mathsf{g}\|_p^p)^{\frac{1}{p}}. \;■$$

It is a routine verification to see that $\|.\|_{1,p}$ defines a norm on $W^{1,p}(a,b)$ and this is left to the reader. Thus, $W^{1,p}(a,b)$ is a normed linear space.

Example 6.4.1 Let $f \in \mathcal{C}^1[a,b]$. Clearly $\mathsf{f} \in L^p(a,b)$. If f' denotes the derivative of f, then $f' \in \mathcal{C}[a,b]$ and so $\mathsf{f}' \in L^p(a,b)$ as well. Further, if $\varphi \in \mathcal{D}(a,b)$, then since $\varphi(a) = \varphi(b) = 0$, we have, by integration by parts,

$$\int_a^b f\varphi' \, dx \;=\; -\int_a^b f'\varphi \, dx.$$

Thus, $\mathsf{f} \in W^{1,p}(a,b)$ and it satisfies (6.4.1) with $g = f'$. ■

By analogy with the preceding example, if $\mathsf{f} \in W^{1,p}(a,b)$, and if g is the associated function as in (6.4.1), then we denote g by f'. In particular, we have

$$\|\mathsf{f}\|_{1,p} = (\|\mathsf{f}\|_p^p + \|\mathsf{f}'\|_p^p)^{\frac{1}{p}}.$$

In the literature, f' is known as the *generalised* or *distributional derivative* of f.

Proposition 6.4.1 *Let $1 \leq p < \infty$ and let $(a,b) \subset \mathbb{R}^n$ be a finite interval. Then, $W^{1,p}(a,b)$ is a Banach space.*

Proof: We just need to prove the completeness. Let $\{\mathsf{f}_n\}$ be a Cauchy sequence in $W^{1,p}(a,b)$. Then $\{\mathsf{f}_n\}$ and $\{\mathsf{f}'_n\}$ are both Cauchy sequences in $L^p(a,b)$. Let $\mathsf{f}_n \to \mathsf{f}$ and $\mathsf{f}'_n \to g$ in $L^p(a,b)$. Now, if $\varphi \in \mathcal{D}(a,b)$, we have

$$\int_a^b f_n\varphi'\,dx = -\int_a^b f'_n\varphi\,dx$$

for all n. Passing to the limit as $n \to \infty$, we deduce that the pair (f,g) satisfies (6.4.1). Thus $\mathsf{f} \in W^{1,p}(a,b)$ and $\mathsf{f}' = \mathsf{g}$. Further, it follows that $\mathsf{f}_n \to \mathsf{f}$ in $W^{1,p}(a,b)$. This completes the proof. ■

Proposition 6.4.2 *The space $W^{1,p}(a,b)$ is reflexive if $1 < p < \infty$ and separable if $1 \leq p < \infty$.*

Proof: Since the space $L^p(a,b)$ is reflexive if $1 < p < \infty$, so is the space $(L^p(a,b))^2$ (why?). Similarly, $(L^p(a,b))^2$ is separable if $1 \leq p < \infty$. Now, the space $W^{1,p}(a,b)$ is isometric to a subspace of $(L^p(a,b))^2$ via the mapping $\mathsf{f} \mapsto (\mathsf{f},\mathsf{f}')$. Since $W^{1,p}(a,b)$ is complete, the image is a closed subspace of $(L^p(a,b))^2$ and so it inherits the reflexivity and separability properties from that space. This completes the proof. ■

We will now study some finer properties of these Sobolev spaces.

Lemma 6.4.2 *Let $\varphi \in \mathcal{D}(a,b)$. Then, there exists $\psi \in \mathcal{D}(a,b)$ such that $\psi' = \varphi$ if, and only if,*

$$\int_a^b \varphi(t)\,dt = 0.$$

Proof: Assume that $\varphi = \psi'$ for some $\psi \in \mathcal{D}(a,b)$. Then, since $\psi(a) = \psi(b) = 0$, it follows that

$$\int_a^b \varphi(t)\ dt \;=\; \int_a^b \psi'(t)\ dt \;=\; \psi(b) - \psi(a) \;=\; 0.$$

Conversely, let $\int_a^b \varphi(t)\ dt = 0$. Let the support of φ be contained in $[c,d] \subset (a,b)$. Now, define

$$\psi(t) \;=\; \int_a^t \varphi(s)\ ds.$$

Clearly ψ is infinitely differentiable since $\psi' = \varphi$. Further, ψ vanishes on the interval (a,c) and, by hypothesis, on the interval (d,b) as well. Thus the support of ψ is also contained in $[c,d]$ and so $\psi \in \mathcal{D}(a,b)$. This completes the proof. ■

Corollary 6.4.1 *Let* f $\in L^p(a,b)$ *where* $1 \le p < \infty$. *Assume that*

$$\int_a^b f\varphi'\ dx \;=\; 0$$

for all $\varphi \in \mathcal{D}(a,b)$. *Then* f *is equal to a constant almost everywhere in* (a,b).

Proof: Choose $\varphi_0 \in \mathcal{D}(a,b)$ such that $\int_a^b \varphi_0(t)\ dt = 1$. Let $\varphi \in \mathcal{D}(a,b)$ be an arbitrary element. Set

$$\phi \;=\; \varphi - \left(\int_a^b \varphi(t)\ dt\right)\varphi_0.$$

Then $\int_a^b \phi(t)\ dt \;=\; 0$ and so $\phi \;=\; \psi'$ for some $\psi \;\in\; \mathcal{D}(a,b)$. Thus, $\int_a^b f\phi\ dt = 0$ which yields

$$\int_a^b f\varphi \;=\; \int_a^b \varphi\ dt. \int_a^b f\varphi_0\ dt.$$

Setting $c = \int_a^b f\varphi_0\ dt$, we get

$$\int_a^b (f-c)\varphi\ dt \;=\; 0$$

for all $\varphi \in \mathcal{D}(a,b)$, whence, by Proposition 6.3.3, it follows that $f(x) = c$ almost everywhere in (a,b). This completes the proof. ■

Remark 6.4.1 If φ_1 were another function in $\mathcal{D}(a,b)$ such that $\int_a^b \varphi_1(t)dt = 1$, then since $\int_a^b(\varphi_0 - \varphi_1)\, dt = 0$, it follows that $\varphi_0 - \varphi_1 = \psi'$ for some $\psi \in \mathcal{D}(a,b)$. Therefore, by hypothesis, $\int_a^b f(\varphi_0 - \varphi_1)\, dt = 0$. Thus, the constant c defined in the above proof does not depend on the choice of the function φ_0 whose integral is unity. ■

Let us denote by $\mathcal{C}^\infty[a,b]$ the space of all functions which are infinitely differentiable in the open interval (a,b) and such that the functions and all their derivatives possess continuous extensions to $[a,b]$.

Proposition 6.4.3 *Let* $1 \le p < \infty$. *Then* $\mathcal{C}^\infty[a,b]$ *is dense in* $W^{1,p}(a,b)$.

Proof: It is clear that if $f \in \mathcal{C}^\infty[a,b]$, then $\mathsf{f} \in W^{1,p}(a,b)$ and its distributional derivative is just its classical derivative. Now, let $\mathsf{f} \in W^{1,p}(a,b)$. Since $\mathsf{f}' \in L^p(a,b)$, choose $\varphi_n \in \mathcal{D}(a,b)$ such that $\varphi_n \to \mathsf{f}'$ in $L^p(a,b)$. Define

$$\psi_n(x) = \int_a^x \varphi_n(t)\, dt.$$

Then $\psi_n \in \mathcal{C}^\infty[a,b]$. Further, for $x \in [a,b]$,

$$|\psi_n(x) - \psi_m(x)| \le \int_a^b |\varphi_n(t) - \varphi_m(t)|\, dt \le (b-a)^{\frac{1}{p^*}} \|\varphi_n - \varphi_m\|_p$$

by Hölder's inequality. Thus,

$$\|\psi_n - \psi_m\|_p \le (b-a)\|\varphi_n - \varphi_m\|_p.$$

It then follows that $\{\psi_n\}$ is Cauchy in $L^p(a,b)$ (since $\{\varphi_n\}$is Cauchy) and so let $\psi_n \to \mathsf{h}$ in $L^p(a,b)$. Since $\psi_n' = \varphi_n$, it is now easy to verify that $\mathsf{h} \in W^{1,p}(a,b)$ and that $\mathsf{h}' = \mathsf{f}'$. By the preceding corollary, it follows that $\mathsf{f} - \mathsf{h}$ is equal to a constant, say c. Thus, if we set $\chi_n = \psi_n + c$, then $\chi_n \in \mathcal{C}^\infty[a,b]$, $\chi_n \to \mathsf{f}$ in $L^p(a,b)$ and $\chi_n' \to \mathsf{f}'$ in $L^p(a,b)$. This completes the proof. ■

We now briefly digress to recall some facts about absolutely continuous functions.

Definition 6.4.2 *A function* $f : [a,b] \to \mathbb{R}$ *is said to be* **absolutely continuous** *on* $[a,b]$ *if, for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* *whenever we have a* finite *collection of* disjoint *intervals* $\{(x_i, x_i')\}_{i=1}^n$ *contained in* (a,b) *satisfying*

$$\sum_{i=1}^n (x_i' - x_i) \ < \ \delta,$$

we have

$$\sum_{i=1}^n |f(x_i') - f(x_i)| \ < \ \varepsilon. \ \blacksquare$$

Clearly, any absolutely continuous function is uniformly continuous. It can also be shown (cf. Royden [5]) that an absolutely continuous function is differentiable almost everywhere and that its derivative is an integrable function. The following two results are very important (cf. Royden [5]).

Theorem 6.4.1 *A function $f : [a,b] \to \mathbb{R}$ can be expressed as an indefinite integral of an integrable function if, and only if, it is absolutely continuous. In this case we have*

$$f(x) \ = \ f(a) + \int_a^x f'(t) \ dt. \ \blacksquare$$

Theorem 6.4.2 *(**Integration by parts**) Let f and g be absolutely continuous functions on $[a,b]$. Then*

$$\int_a^b f(t)g'(t) \ dt \ = \ f(b)g(b) - f(a)g(a) - \int_a^b f'(t)g(t) \ dt. \ \blacksquare$$

If $f \in C^1[a,b]$, then it is absolutely continuous. In particular, if $\varphi \in \mathcal{D}(a,b)$, it is absolutely continuous. Consequently, by virtue of intergation by parts, it follows that if f is absolutely continuous on $[a,b]$ and if $\varphi \in \mathcal{D}(a,b)$, then

$$\int_a^b f\varphi' \ dt \ = \ -\int_a^b f'\varphi \ dt$$

so that the distributional derivative of f is f'.

Proposition 6.4.4 *Let $1 \le p < \infty$. Let $\mathsf{f} \in W^{1,p}(a,b)$. Then f is absolutely continuous, i.e. f is equal, almost everywhere, to an absolutely continuous function.*

Proof: Let us define

$$u(x) = \int_a^x f'(t)\, dt.$$

Since $f' \in L^p(a,b)$ and since $p \geq 1$, it follows that f' is integrable on (a,b) and so u is an absolutely continuous function. Further, since integration by parts is valid for absolutely continuous functions, it follows that for all $\varphi \in \mathcal{D}(a,b)$, we have

$$\int_a^b u\varphi'\, dx = -\int_a^b f'\varphi\, dx.$$

Thus $u \in W^{1,p}(a,b)$ and $u' = f'$. Then, as before, it follows that $f - u$ is equal to a constant almost everywhere. Thus $f(x) = u(x) + c$ almost everywhere in x and the latter function is absolutely continuous. ■

The above proposition states that $W^{1,p}(a,b)$ consists of absolutely continuous functions (upto equality almost everywhere). In particular, we can say that $W^{1,p}(a,b)$ is contained in $\mathcal{C}[a,b]$, *i.e.* every element of $W^{1,p}(a,b)$ is represented by means of an (absolutely) continuous function. Such a representative must be unique, for, if two continuous functions are equal almost everywhere, then they are equal everywhere (why?).

Theorem 6.4.3 (**Sobolev's Theorem**) *The inclusion map from $W^{1,p}(a,b)$ into $\mathcal{C}[a,b]$ is continuous.*

Proof: Let f_n, f be in $W^{1,p}(a,b)$ with absolutely continuous representatives f_n, f. Assume that $f_n \to f$ in $W^{1,p}(a,b)$. Then $\|f_n - f\|_p \to 0$ and $\|f_n' - f'\|_p \to 0$. Now, by absolute continuity, we have

$$f_n(x) = f_n(a) + \int_a^x f_n'(t)\, dt \tag{6.4.2}$$

and

$$f(x) = f(a) + \int_a^x f'(t)\, dt. \tag{6.4.3}$$

We claim that $\{f_n(a)\}$ is Cauchy. If not, there exists $\varepsilon > 0$ such that, for every N, there exist $m, n \geq N$ satisfying $|f_m(a) - f_n(a)| \geq \varepsilon$. Then, it follows from (6.4.2) that

$$|f_m(x) - f_n(x)| \geq \varepsilon - \|f_m' - f_n'\|_p (x-a)^{\frac{1}{p^*}}$$

by an application of Hölder's inequality. Choose N large enough such that, for all $n, m \geq N$, we have

$$\|f_m' - f_n'\|_p (b-a)^{\frac{1}{p^*}} < \frac{\varepsilon}{2}.$$

Then, for all $x \in (a,b)$ we have

$$|f_m(x) - f_n(x)| \ \geq \ \frac{\varepsilon}{2}$$

whence it would follow that

$$\|f_m - f_n\|_p \geq (b-a)^{\frac{1}{p}} \frac{\varepsilon}{2} \ > \ 0$$

whch contradicts the fact the $\{\mathsf{f}_n\}$ is cauchy in $L^p(a,b)$.

Thus $\{f_n(a)\}$ is Cauchy and now, for any $x \in [a,b]$,

$$|f_m(x) - f_n(x)| \ \leq \ |f_m(a) - f_n(a)| + \|f_m' - f_n'\|_p (b-a)^{\frac{1}{p^*}}$$

by another application of (6.4.2) and Hölder's inequality. This shows that $\{f_n\}$ is uniformly Cauchy and so it converges to a continuous function f on $[a,b]$. But since $\|f_n - f\|_p \to 0$, it follows that (cf. Corollary 6.1.1), at least for a subsequence, we have $f_{n_k}(x) \to f(x)$ almost everywhere, from which we deduce that $f \equiv \widetilde{f}$. Thus, $f_n \to f$ in $\mathcal{C}[a,b]$ which completes the proof. ■

Theorem 6.4.4 (**Rellich's Theorem**) *The unit ball in $W^{1,p}(a,b)$ is relatively compact in $L^p(a,b)$.*

Proof: The inclusion map $W^{1,p}(a,b) \subset L^p(a,b)$ is the composition of the following inclusion maps:

$$W^{1,p}(a,b) \ \subset \ \mathcal{C}[a,b] \ \subset \ L^p(a,b).$$

The first inclusion above is continuous by the preceding theorem. The space $\mathcal{C}[a,b]$ is a subspace of $L^\infty(a,b)$ and the 'sup-norm' is the same as $\|.\|_\infty$. Now it follows that the second inclusion is also continuous by Proposition 6.1.3.

Let B be the unit ball in $W^{1,p}(a,b)$. Thus, if $\mathsf{f} \in B$, then

$$\|\mathsf{f}\|_p^p + \|\mathsf{f}'\|_p^p \ \leq \ 1.$$

Then, by the preceding theorem, it follows that B is bounded in $\mathcal{C}[a,b]$ as well, since the inclusion map is continuous (cf. Proposition 2.3.1 (iv)).

Further, let $x, y \in [a,b]$. Assume, without loss of generality, that $x \le y$. Then

$$|f(x) - f(y)| \;\le\; \left|\int_x^y f'(t)\, dt\right| \;\le\; \|f'\|_p |y-x|^{\frac{1}{p^*}} \;\le\; |y-x|^{\frac{1}{p^*}}.$$

It now follows immediately that B is equicontinuous as well since for $\varepsilon > 0$, if we choose $\delta < \varepsilon^{p^*}$, then $|x-y| < \delta$ implies that $|f(x)-f(y)| < \varepsilon$ for all $\mathsf{f} \in B$. Thus, by the theorem of Ascoli, it follows that B is relatively compact in $\mathcal{C}[a,b]$.

Thus, any sequence in B will have a subsequence which is convergent in $\mathcal{C}[a,b]$, which will also, *a fortiori*, converge in $L^p(a,b)$. This proves that B is relatively compact in $L^p(a,b)$. ■

Definition 6.4.3 *Let $(a,b) \subset \mathbb{R}$ be a finite interval and let $1 \le p < \infty$. The closure of $\mathcal{D}(a,b)$ in $W^{1,p}(a,b)$ is denoted $W_0^{1,p}(a,b)$.* ■

Theorem 6.4.5 *Let $\mathsf{f} \in W^{1,p}(a,b)$ with f absolutely continuous. Then $\mathsf{f} \in W_0^{1,p}(a,b)$ if, and only if, $f(a) = f(b) = 0$.*

Proof: Let $\mathsf{f} \in W_0^{1,p}(a,b)$. Then there exists a sequence $\{\varphi_n\}$ in $\mathcal{D}(a,b)$ such that $\varphi_n \to \mathsf{f}$ in $W^{1,p}(a,b)$. Then $\varphi_n \to f$ uniformly on $[a,b]$ and so it follows immediately that $f(a) = f(b) = 0$.

Conversely, let $f(a) = f(b) = 0$. Then (cf. Theorem 6.4.1) we have

$$f(x) \;=\; \int_a^x f'(t)\, dt$$

and so it follows that $\int_a^b f'(t)\, dt = 0$. Let $\varphi_n \in \mathcal{D}(a,b)$ such that $\|\varphi_n - f'\|_p \to 0$. Then

$$\left|\int_a^b \varphi_n\, dt - \int_a^b f'\, dt\right| \;\le\; \|\varphi_n - f'\|_p (b-a)^{\frac{1}{p^*}} \;\to\; 0$$

and so

$$\int_a^b \varphi_n\, dt \;\to\; 0.$$

Let $\varphi_0 \in \mathcal{D}(a,b)$ such that $\int_a^b \varphi_0\, dt = 1$. Then if

$$\psi_n = \varphi_n - \left(\int_a^b \varphi_n\, dt\right)\varphi_0,$$

we also have that $\|\psi_n - f'\|_p \to 0$ and $\int_a^b \psi_n \, dt = 0$. Thus $\psi_n = \chi_n'$ where $\chi_n \in \mathcal{D}(a,b)$ as well (cf. Lemma 6.4.2). Since

$$\chi_n(x) = \int_a^x \psi_n \, dt,$$

it follows that χ_n converges to f uniformly and so $\|\chi_n - f\|_p \to 0$ as well. Thus $\chi_n \in \mathcal{D}(a,b)$ and $\chi_n \to \mathsf{f}$ in $W^{1,p}(a,b)$. This proves that $\mathsf{f} \in W_0^{1,p}(a,b)$. ■

Notice that if $f \equiv 1$, then $f' \equiv 0$ so that, in $W^{1,p}(a,b)$, the map $\mathsf{f} \mapsto \|\mathsf{f}'\|_p$ does not define a norm, but only a *semi-norm*, *i.e.* while $\|\mathsf{f}'\|_p = 0$ does not imply that $\mathsf{f} = \mathbf{0}$, all other properties of a norm are satisfied. However, in the space $W_0^{1,p}(a,b)$, we have the following result.

Theorem 6.4.6 *(***Poincaré's Inequality***) Let $\mathsf{f} \in W_0^{1,p}(a,b)$. Then*

$$\|\mathsf{f}\|_p \le (b-a)\|\mathsf{f}'\|_p. \tag{6.4.4}$$

Thus the function $\mathsf{f} \mapsto \|\mathsf{f}'\|_p$ defines a norm on $W_0^{1,p}(a,b)$ equivalent to the usual norm on this space.

Proof: Let f be absolutely continuous and represent f. If $\mathsf{f} \in W_0^{1,p}(a,b)$, then, since $f(a) = 0$, we have

$$f(x) = \int_a^x f'(t) \, dt.$$

Then, by Hölder's inequality, we have

$$|f(x)| \le \|f'\|_p (b-a)^{\frac{1}{p^*}}.$$

Thus,

$$\|f\|_p \le \|f'\|_p (b-a)^{\frac{1}{p}+\frac{1}{p^*}} = (b-a)\|f'\|_p$$

which proves (6.4.4).

In particular, if $\|\mathsf{f}'\|_p = 0$, it follows that $\|\mathsf{f}\|_p = 0$ and so $\mathsf{f} = \mathbf{0}$ in $W_0^{1,p}(a,b)$. The other properties of a norm are easily verified. Thus we have two norms on $W_0^{1,p}(a,b)$:

$$\|\mathsf{f}\|_{1,p} \text{ and } |\mathsf{f}|_{1,p} \overset{\text{def}}{=} \|\mathsf{f}'\|_p.$$

Clearly,

$$|\mathsf{f}|_{1,p} \le \|\mathsf{f}\|_{1,p} \le [(b-a)^p + 1]^{\frac{1}{p}} |\mathsf{f}|_{1,p}.$$

Thus the two norms are equivalent. ■

Sobolev spaces can also be defined when $p = \infty$. The definition can also be extended to cover functions defined on arbitrary open sets $\Omega \subset \mathbb{R}^N$. It is also possible to define 'higher order distributional derivatives' and define Sobolev spaces $W^{m,p}(\Omega)$, $m \in \mathbb{N}$, based on these derivatives. All these spaces have properties similar to those proved in this section, with or without additional hypotheses. For a detailed study of Sobolev spaces, see Kesavan [3]. See also the Exercises 6.24-6.26 below.

6.5 Exercises

6.1 Let $(X, \mathcal{S}, \mu)$ be a measure space. let $1 \leq p, q, r < \infty$ such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

If $\mathsf{f} \in L^p(\mu)$ and $\mathsf{g} \in L^q(\mu)$, show that $\mathsf{fg} \in L^r(\mu)$ and that

$$\|\mathsf{fg}\|_r \leq \|\mathsf{f}\|_p \|\mathsf{g}\|_q.$$

6.2 Let $(X, \mathcal{S}, \mu)$ be a measure space and let $1 \leq p < \infty$. Define, for $t > 0$,

$$h_f(t) = \mu(\{|f| > t\}).$$

Show that

$$\|\mathsf{f}\|_p^p = p \int_0^\infty t^{p-1} h_f(t)\, dt.$$

(Hint: Write h_f as an integral over a subset of X and apply Fubini's theorem (cf. Theorem 1.3.5).)

6.3 (a) Let $(X, \mathcal{S}, \mu)$ be a measure space. Let f_n, g_n, f, g be measurable functions such that $f_n \to f$ and $g_n \to g$ almost everywhere in X. Assume further that $|f_n(x)| \leq g_n(x)$ for all $x \in X$ and that

$$\int_X g_n \, d\mu \to \int_X g \, d\mu < \infty$$

as $n \to \infty$. Show that

$$\int_X f_n \, d\mu \to \int_X f \, d\mu$$

as $n \to \infty$. (Hint: Apply Fatou's lemma (cf. Theorem 1.3.2) to $g_n + f_n \geq 0$ and to $g_n - f_n \geq 0$.)
(b) Let $1 \leq p < \infty$. Let f_n and $\mathsf{f} \in L^p(\mu)$ and assume that $f_n(x) \to f(x)$ almost everywhere in X. Show that $\mathsf{f}_n \to \mathsf{f}$ in $L^p(\mu)$ if, and only if, $\|\mathsf{f}_n\|_p \to \|\mathsf{f}\|_p$.

6.4 Let $(X, \mathcal{S}, \mu)$ be a measure space and let $1 \leq p < \infty$. Let $\mathsf{f}_n \to \mathsf{f}$ in $L^p(\mu)$. Let g_n be a sequence of measurable functions converging to a measurable function g almost everywhere in X. Assume further that g_n and g are all uniformly bounded by a constant $M > 0$ in X. Show that $\mathsf{f}_n\mathsf{g}_n \to \mathsf{fg}$ in $L^p(\mu)$.

6.5 Let $(X, \mathcal{S}, \mu)$ be a measure space. A sequence of measurable functions f_n is said to *converge in measure* in X to a measurable function f if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu(\{|f_n - f| \geq \varepsilon\}) \ = \ 0.$$

In this case, we write $f_n \xrightarrow{\mu} f$. If $1 \leq p < \infty$ and if $\mathsf{f}_n \to \mathsf{f}$ in $L^p(\mu)$, show that $f_n \xrightarrow{\mu} f$.

6.6 Let $(X, \mathcal{S}, \mu)$ be a measure space and let $1 < p < \infty$. Let $f : X \times X \to \mathbb{R}$ be such that for almost every $y \in X$, the section f^y (cf. Definition 1.3.7) is p-integrable. Define, for $x \in X$,

$$g(x) \ = \ \int_X f(x,y) \ d\mu(y).$$

Show that $\mathsf{g} \in L^p(\mu)$ and that

$$\|\mathsf{g}\|_p \ \leq \ \int_X \|f^y\|_p d\mu(y).$$

6.7 Let $g \in C_c(\mathbb{R})$. Define $\varphi(g) = g(0)$. Then φ can be extended to a continuous linear functional on $L^\infty(\mathbb{R})$. Show that there does not exist $\mathsf{f} \in L^1(\mathbb{R})$ such that

$$\varphi(g) \ = \ \int_{\mathbb{R}} gf \ dx$$

for all $g \in L^\infty(\mathbb{R})$. (This gives another proof that $L^\infty(\mathbb{R})$ is not reflexive.)

6.8 Let $h \in \mathbb{R}^N$. For a (Lebesgue) measurable function f defined on $\mathbb{R}^N$, define its translation by h by

$$f_h(x) = f(x+h).$$

If $\mathsf{f} \in L^p(\mathbb{R}^N)$, show that $\mathsf{f}_h \in L^p(\mathbb{R}^N)$ and that

$$\|\mathsf{f} - \mathsf{f}_h\|_p \to 0$$

as $h \to \mathbf{0}$ in $\mathbb{R}^N$ for any $1 \le p < \infty$.

6.9 Let

$$f_n = \chi_{[n,n+1]},$$

the characteristic function of the closed interval $[n, n+1]$ for $n \in \mathbb{N}$ (cf. Definition 1.3.5). Then $\{\mathsf{f}_n\}$ is a bounded sequence in $L^1(\mathbb{R})$. Show that it does not have a weakly convergent subsequence. (In view of Theorem 5.4.2, this gives another proof that $L^1(\mathbb{R})$ is not reflexive.)

6.10 Let $f \in C[0,1]$ be such that, for all $n \ge 0$,

$$\int_0^1 x^n f(x)\, dx = 0.$$

Show that $f \equiv 0$.

6.11 (Hardy's inequality) Let $\mathsf{f} \in L^p(0,\infty)$, where $1 < p < \infty$. Define

$$g(x) = \frac{1}{x}\int_0^x f(t)\, dt$$

for $x \in (0,\infty)$. Show that $\mathsf{g} \in L^p(0,\infty)$ and that

$$\|\mathsf{g}\|_p \le \frac{p}{p-1}\|\mathsf{f}\|_p.$$

(Hint: Prove it first for $f \in C_c(0,\infty), f \ge 0$.)

6.12 A function $\varphi : (0,\infty) \to \mathbb{R}$ is said to be a *step function* if

$$\varphi(x) = \sum_{j=1}^{n} \alpha_j \chi_{I_j}(x)$$

where $I_j, 1 \leq j \leq n$ are intervals contained in $(0, \infty)$ and, as usual, χ_E denotes the characteristic function of a set E. Show that step functions in $(0, \infty)$ are dense in $L^1(0, \infty)$.

6.13 (Riemann-Lebesgue lemma) Let h be a bounded and measurable function on $(0, \infty)$ such that

$$\lim_{c \to \infty} \frac{1}{c} \int_0^c h(t)\, dt = 0.$$

(a) Let $f = \chi_{[c,d]}$, where $[c, d] \subset (0, \infty)$. Show that

$$\lim_{\omega \to \infty} \int_0^\infty f(t)h(\omega t)\, dt = 0. \tag{6.5.5}$$

(b) Deduce that (6.5.5) is true for all $\mathsf{f} \in L^1(0, \infty)$.
(c) If $\mathsf{f} \in L^1(a, b)$ where $(a, b) \subset (0, \infty)$, show that

$$\lim_{n \to \infty} \int_a^b f(t) \cos nt\, dt = \lim_{n \to \infty} \int_a^b f(t) \sin nt\, dt = 0.$$

6.14 (a) Let $(a, b) \subset (0, \infty)$ be any finite interval. Let $f_n(t) = \cos nt$ and let $g_n(t) = \sin nt$. Show that $\mathsf{f}_n \rightharpoonup \mathbf{0}$ and $\mathsf{g}_n \rightharpoonup \mathbf{0}$ in $L^p(a, b)$ for any $1 \leq p < \infty$.
(b) What is the weak limit of h_n in $L^p(a, b)$ for $1 \leq p < \infty$ where $h_n(t) = \cos^2 nt$?

6.15 (a) Consider the trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

Show that it can written in the *amplitude-phase* form

$$\frac{a_0}{2} + d_n \cos(nt - \phi_n).$$

Write down the relations between a_n, b_n and d_n, ϕ_n.
(b) (Cantor-Lebesgue theorem) Show that if a trigonometric series as in (a) above converges over a set E whose measure is strictly positive, then $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$. (Hint: Use the amplitude-phase form of the series.)

6.16 If f and g $\in L^1(\mathbb{R}^N)$, show that

$$\mathsf{f} * \mathsf{g} \;=\; \int_{\complement^N} f(y)g(x-y)\;dy$$

is well defined for almost all $x \in \mathbb{R}^N$. Show also that $\mathsf{f} * \mathsf{g} \in L^1(\mathbb{R}^N)$ and that

$$\|\mathsf{f} * \mathsf{g}\|_1 \;\leq\; \|\mathsf{f}\|_1 \|\mathsf{g}\|_1.$$

6.17 Let $\{\rho_\varepsilon\}_{\varepsilon>0}$ be a family of C^∞ functions in $\mathbb{R}^N$ such that for each $\varepsilon > 0$, we have that $\rho_\varepsilon(x) \geq 0$ for all $x \in \mathbb{R}^N$, the support of ρ_ε is contained in the closed ball with centre at the origin and radius ε, and

$$\int_{\mathbb{R}^N} \rho_\varepsilon(x)\;dx \;=\; 1.$$

(a) Let $\varphi \in C_c(\mathbb{R}^N)$. Show that $\rho_\varepsilon * \varphi$ converges uniformly to φ on $\mathbb{R}^N$ as $\varepsilon \to 0$.
(b) Deduce that, if $u \in L^p(\mathbb{R}^N)$, then $\rho_\varepsilon * u$ converges to u in $L^p(\mathbb{R}^N)$ as $\varepsilon \to 0$.

6.18 Let $\{\mathsf{f}_n\}$ be a bounded sequence in $L^p(a,b)$, where (a,b) is an open interval in $\mathbb{R}$ and $1 \leq p \leq \infty$. Show that $\mathsf{f}_n \rightharpoonup \mathsf{f}$ in $L^p(a,b)$ when $1 \leq p < \infty$ and $\mathsf{f}_n \overset{*}{\rightharpoonup} \mathsf{f}$ in $L^\infty(a,b)$ if, and only if, for every $\varphi \in \mathcal{D}(a,b)$, we have

$$\int_a^b f_n\varphi\;dx \to \int_a^b f\varphi\;dx.$$

6.19 Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$. Define the sequence $\{f_n\}$ as follows. Let $f_n(x) = f(nx)$ on $[0, \frac{1}{n}]$ and extend this periodically to each subinterval $[\frac{k-1}{n}, \frac{k}{n}]$ for $2 \leq k \leq n$. Let $m = \int_0^1 f(t)\;dt$. Show that $\mathsf{f}_n \rightharpoonup \mathsf{f}$ in $L^p(0,1)$ for $1 \leq p < \infty$ and that $\mathsf{f}_n \overset{*}{\rightharpoonup} \mathsf{f}$ in $L^\infty(0,1)$, where $f(t) = m$ for all $t \in [0,1]$.

6.20 Let $(a,b) \subset \mathbb{R}$ be a finite interval and let $f : [a,b] \to \mathbb{R}$ be a Lipschitz continuous function *i.e.* there exists $K > 0$ such that for all $x, y \in [a,b]$, we have

$$|f(x) - f(y)| \;\leq\; K|x-y|.$$

Show that $\mathsf{f} \in W^{1,p}(a,b)$ for all $1 \leq p < \infty$.

6.21 Let $a < c < b$ in $\mathbb{R}$. Let $f : [a,b] \to \mathbb{R}$ be continuous. Assume that $\mathsf{f} \in W^{1,p}(a,c)$ and that $\mathsf{f} \in W^{1,p}(b,c)$. Show that $\mathsf{f} \in W^{1,p}(a,b)$.

6.22 Let $f : [-1,1] \to \mathbb{R}$ be given by

$$f(x) \;=\; \begin{cases} 0, & \text{if } x \in [-1,0) \\ 1, & \text{if } x \in [0,1] \end{cases}$$

Show that $\mathsf{f} \notin W^{1,p}(-1,1)$ for $1 \le p < \infty$. (Thus, continuity is essential in the previous exercise.)

6.23 (a) Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous and assume that

$$\int_a^b f(t)\,dt \;=\; 0.$$

Let $1 \le p < \infty$. Show that

$$|f(x)| \;\le\; 2(b-a)^{\frac{1}{p^*}}\|f'\|_p$$

for all $x \in [a,b]$.
(b) (*Poincaré-Wirtinger Inequality*) Deduce that, for all $\mathsf{f} \in W^{1,p}(a,b)$ such that $\int_a^b f(t)\,dt = 0$, we have

$$\|\mathsf{f}\|_p \;\le\; 2(b-a)\|\mathsf{f}'\|_p.$$

6.24 (a) Define $W^{1,p}(\mathbb{R})$ exactly as in Definition 6.4.1, with $\mathbb{R}$ replacing the interval (a,b) in that definition as well as in relation (6.4.1). Let ζ be a $\mathcal{C}^\infty$ function on $\mathbb{R}$ with compact support. Show that, if $\mathsf{f} \in W^{1,p}(\mathbb{R})$, then $\zeta\mathsf{f} \in W^{1,p}(\mathbb{R})$ as well, where $(\zeta f)(x) = \zeta(x)f(x)$.
(b) Let ζ be a $\mathcal{C}^\infty$ function on $\mathbb{R}$ with compact support contained in $[-2,2]$ such that $0 \le \zeta(x) \le 1$ for all $x \in \mathbb{R}$ and such that $\zeta \equiv 1$ on $[-1,1]$. Define $\zeta_m(x) = \zeta(x/m)$ for all $x \in \mathbb{R}$. Show that if $u \in W^{1,p}(\mathbb{R})$ for $1 \le p < \infty$, then $\zeta_m u \to u$ in $W^{1,p}(\mathbb{R})$ as $m \to \infty$.

6.25 Let $(a,b) \subset \mathbb{R}$ be a finite open interval. Let $m > 1$ be a positive integer. Define, for $1 \le m < \infty$,

$$W^{m,p}(a,b) \;=\; \left\{ \mathsf{f} \in L^p(a,b) \;\middle|\; \begin{array}{c} \text{there exist } \mathsf{g}_i \in L^p(a,b),\ 1 \le i \le m \\ \text{such that} \\ \int_a^b f\frac{d^i\varphi}{dx^i}\,dx \;=\; (-1)^i\int_a^b g_i\varphi\,dx \\ \text{for all } \varphi \in \mathcal{D}(a,b) \end{array} \right\}.$$

The functions g_i are called the generalized successive derivatives of f and we denote $\mathsf{f}^{(i)} = g_i$. Define

$$\|\mathsf{f}\|_{m,p} = \left(\|f\|_p^p + \sum_{i=1}^{m} \|\mathsf{f}^{(i)}\|_p^p \right)^{\frac{1}{p}}$$

for $\mathsf{f} \in W^{m,p}(a,b)$.
(a)Show that $\|.\|_{m,p}$ defines a norm on $W^{m,p}(a,b)$ which makes it into a Banach space which is separable if $1 \leq p < \infty$ and reflexive if $1 < p < \infty$.
(b) Show that if $\mathsf{f} \in W^{m,p}(a,b)$, then $f \in C^{m-1}[a,b]$.

6.26 Let $W_0^{m,p}(a,b)$ denote the closure of $\mathcal{D}(a,b)$ in $W^{m,p}(a,b)$.
(a) Show that $\mathsf{f} \in W^{m,p}(a,b)$ belongs to $W_0^{m,p}(a,b)$ if, and only if, $f(a) = f(b) = 0$ and $f^{(i)}(a) = f^{(i)}(b) = 0$ for all $1 \leq i \leq m-1$.
(b) Show that

$$\mathsf{f} \mapsto |\mathsf{f}|_{m,p} \stackrel{\text{def}}{=} \|\mathsf{f}^{(m)}\|_p$$

defines a norm on $W_0^{m,p}(a,b)$ which is equivalent to the usual norm $\|\cdot\|_{m,p}$.